

Problem 1

$$a) \frac{\partial^2}{\partial x^2}(e^x \cos y) + \frac{\partial^2}{\partial y^2}(e^x \cos y) = e^x \cos y - e^x \cos y \\ = 0.$$

$$b) \frac{\partial^2}{\partial y \partial x}(e^x \cos y) = \frac{\partial}{\partial y} e^x \cos y = -e^x \sin y$$

$$\frac{\partial^2}{\partial x \partial y}(e^x \cos y) = \frac{\partial}{\partial x} e^x (-\sin y) = -e^x \sin y$$

$$c) f(r, \theta) = -\log(r^2)$$

$$\frac{\partial^2}{\partial r^2}(-\log r^2) + \frac{1}{r} \frac{\partial}{\partial r}(-\log r^2) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} f$$

$$= \frac{\partial}{\partial r} \left(-\frac{2r}{r^2} \right) + \frac{1}{r} \left(-\frac{2r}{r^2} \right)$$

$$= \frac{2}{r^2} - \frac{2}{r^2}$$

$$= 0.$$

Problem 2:

a) 4.5.1

$$z = x e^{-y}, \quad x = \cosh t, \quad y = \cosh t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= e^{-y} \cdot \sinh t + (-x e^{-y}) (-\sinh t)$$

$$= e^{-\cosh t} (\sinh t + \cosh t \sinh t).$$

b) 4.5.7

$$c = \sin(a-b), \quad b = ae^{2a} \quad \left] \rightarrow \frac{db}{da} = e^{2a} + 2ae^{2a} \right.$$
$$\frac{dc}{da} = \cos(a-b) \cdot \left(1 - \frac{db}{da}\right)$$
$$= \cos(a-b) \cdot (1 - (e^{2a} + 2ae^{2a}))$$

Problem 3

4.6.4:

$xe^y = ye^x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $y \neq 1$.

$$\frac{d}{dx} (||) \Rightarrow \underbrace{1 \cdot e^y + x e^y \frac{dy}{dx}}_{(*)} = \frac{dy}{dx} e^x + y e^x$$

$$\frac{d}{dx} (\quad)$$

$$\Rightarrow e^y \frac{dy}{dx} + 1 \cdot e^y \frac{dy}{dx} + x e^y \left(\frac{dy}{dx} \right)^2 + x e^y \frac{d^2y}{dx^2} \stackrel{(*)}{=} \frac{d^2y}{dx^2} e^x + \frac{dy}{dx} e^x + \frac{dy}{dx} e^x + y e^x$$

$$(*): \quad \frac{dy}{dx} = \frac{e^y - y e^x}{x e^y - e^x}$$

Then solve $(*)$ for $\frac{d^2y}{dx^2}$ with a pile of algebra.

Problem 4:

a) 4.7.2

$P = r \cos t$, $\leftarrow P(r, t)$ is $P: \mathbb{R}^2 \rightarrow \mathbb{R}$.

(*) $[r \sin t - 2te^r = 0]$ $\leftarrow$ this is 1 eqn in $\mathbb{R}^2$,
so we may "locally" say
 $r(t)$.

2 degrees freedom - 1 eqn = 1 degree freedom

$\Rightarrow P: \mathbb{R} \rightarrow \mathbb{R}$ is

$\Rightarrow P = r(t) \cos t$

$$\frac{dP}{dt} = \frac{dr}{dt} \cos t + r(-\sin t)$$

$$\frac{d}{dt} (*) = 0 = \frac{dr}{dt} \sin t + r \cos t - (2e^r + 2te^r \frac{dr}{dt})$$

so
$$\frac{dr}{dt} = \frac{-r \cos t + 2e^r}{\sin t - 2te^r}$$

$$\Rightarrow \left[\frac{dP}{dt} = \left(\frac{-r \cos t + 2e^r}{\sin t - 2te^r} \right) \cos t + r(-\sin t) \right]$$

Ex:
 $x^2 + y^2 = 1$



in this
"local" region,
can say y is
a function of x ,
 $y(x)$,
(Not true on whole
circle!)

b) 4.7.7 $\left(\frac{\partial x}{\partial \theta} \right)_r$ means derivative of x in direction of θ ,
keeping r fixed. So, need $x = x(r, \theta)$

$x = r \cos \theta$ so $\left(\frac{\partial x}{\partial \theta} \right)_r = -r \sin \theta$, $\left(\frac{\partial y}{\partial \theta} \right)_r = r \cos \theta$
 $y = r \sin \theta$

Now, to compute $\left(\frac{\partial y}{\partial \theta} \right)_x$, need $y(\theta, x)$.

$\Rightarrow y = \frac{x}{\cos \theta} \cdot \sin \theta = x \tan \theta$

$$\left(\frac{\partial y}{\partial \theta} \right)_x = x \sec^2 \theta = \frac{r \cos \theta}{\cos^2 \theta} = \frac{r}{\cos \theta}$$

To compute $\left(\frac{\partial \theta}{\partial y}\right)_x$, need $\theta = \theta(x, y)$.

$$\text{well, } \frac{\sin \theta}{\cos \theta} = \frac{y}{x}$$

$$\Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

$$\begin{aligned}\text{So } \left(\frac{\partial \theta}{\partial y}\right)_x &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{1 + \left(\frac{\sin \theta}{\cos \theta}\right)^2} \cdot \frac{1}{r \cos \theta} \\ &= \frac{1}{r} \left(\frac{1}{\cos \theta + \frac{\sin^2 \theta}{\cos \theta}} \right) \\ &= \frac{1}{r} \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \\ &= \frac{\cos \theta}{r}\end{aligned}$$

$\left(\frac{\partial \theta}{\partial y}\right)_x$ and $\left(\frac{\partial y}{\partial \theta}\right)_x$ are reciprocals only when the variable held constant is the same.

Problem 5

4.8.6

$$f(x, y) = x^3 - y^3 - 2xy + 2$$

$$\text{Compute: } \nabla f = \begin{bmatrix} 3x^2 - 2y \\ -3y^2 - 2x \end{bmatrix} \quad \downarrow \quad = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x^2 = 2y$$

$$3y^2 = -2x$$

• Easy soln at $(x, y) = \vec{0}$. Are there others?

$$x = -\frac{3}{2}y^2 \Rightarrow 3\left(-\frac{3}{2}y^2\right)^2 - 2y = 0$$

$$3\left(-\frac{3}{2}y^2\right)^2 - 2y = 0$$

$$\frac{27}{4}y^4 - 2y = 0$$

$$27y^4 - 8y = 0$$

$$\Rightarrow (27y^3 - 8) = 0 \quad \rightarrow y = 0$$

$$y^3 = \frac{8}{27}, \text{ so}$$

$$y = \frac{2}{3}, \quad x = -\frac{3}{2}\left(\frac{2}{3}\right)^2 = -\frac{2}{3}$$

of course, we can flip signs, so

critical points:

$$(0,0), \left(-\frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}\right)$$

Now: compute

$$f_{xx} = 6x$$

$$f_{yy} = -6y$$

$$f_{xy} = f_{yx} = -2$$

• $(0,0)$: Neither since $f_{xx} = 0$, $f_{yy} = 0$, but $f_{xy}^2 = 4$.

• $\left(\frac{2}{3}, -\frac{2}{3}\right)$:
 $f_{xx} = 6 \cdot \frac{2}{3} = 4 \Rightarrow f_{xx} + f_{yy} < f_{xy}^2$,
 $f_{yy} = 6 \cdot -\frac{2}{3} = -4$
 $f_{xy} = -2$
 local max.

$$\left(-\frac{2}{3}, \frac{2}{3}\right)$$

$$f_{xx} = 6 \cdot \frac{-2}{3} = -4$$

$$f_{yy} = 6 \cdot \frac{2}{3} = 4$$

$$f_{xy} = -2$$

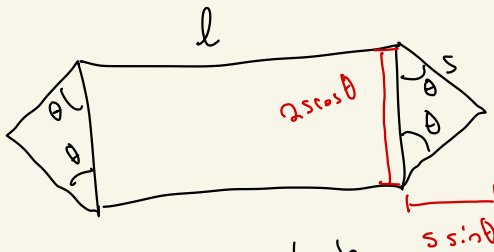
 $\Rightarrow$

$$f_{xx} + f_{yy} < f_{xy},$$

local max.

Problem 6

a) 4.4.1



Maximize: $A = 2 \text{ Triangle} + \text{rectangle}$
 $= 2s^2 \sin \theta \cos \theta + 2sl \cos \theta$
 $= s^2 \sin 2\theta + 2sl \cos \theta.$

$$A = A(s, l, \theta)$$

$$P = P(s, l, \theta)$$

$$\theta = \frac{\pi}{6}$$

Constraint: $P = 2l + 4s$

Evidently by looking at diagram,

$$s, l > 0 \quad \text{and} \quad 0 < \theta < \frac{\pi}{2}.$$

Lagrange:

$$\nabla A = \lambda \nabla P$$

$$\frac{\partial}{\partial s} \quad 2s \sin 2\theta + 2l \cos \theta = \lambda 4$$

$$\frac{\partial}{\partial l} \quad 2s \cos \theta = \lambda 2$$

$$\frac{\partial}{\partial \theta} \quad 2s^2 \cos 2\theta - 2sl \sin \theta = 0$$

Since s, l, θ non-zero, $(\frac{\partial}{\partial \theta})$ eqn gives

$$s \cos 2\theta = l \sin \theta$$

$$\Rightarrow \quad s = l \frac{\sin \theta}{\cos 2\theta}$$

$$\text{so } 2 \left(l \frac{\sin \theta}{\cos 2\theta} \right) \sin 2\theta + 2l \cos \theta = \lambda 4,$$

$$2l \frac{\sin \theta}{\cos 2\theta} \sin 2\theta + 2l \cos \theta = \lambda 4 = 4 \left(l \frac{\sin \theta}{\cos 2\theta} \right) \cos \theta$$

$$\Rightarrow \sin \theta \sin 2\theta + \cos 2\theta \cos \theta = 2 \sin \theta \cos \theta$$

which gives (after recalling trig)

$$\sin \theta 2 \sin \theta \cos \theta + \cos 2\theta \cos \theta = 2 \sin \theta \cos \theta$$

$$(2 \sin^2 \theta + \cos 2\theta) \cos \theta = (2 \sin \theta) \cos \theta$$

Since $\theta < \pi/2$, $\cos \theta \neq 0$, so we may divide.

$$2 \sin^2 \theta + \cos 2\theta = 2 \sin \theta$$

$$1 = 2 \sin \theta \Rightarrow \theta = \pi/6$$

$\cos^2 \theta - \sin^2 \theta = \cos 2\theta$
plug in to
find s, l .
It's not too
bad.

b) 4.9.4

We solved this in lecture! So get

$$(x, y, z) = \frac{1}{\sqrt{3}} (2, 3, 5)$$

So the box's side lengths are

$$\frac{4}{\sqrt{3}}, \frac{6}{\sqrt{3}}, \frac{10}{\sqrt{3}}.$$

c) 4.9.11

Want:

$$\min x^2 + y^2 + z^2 = f(x, y, z)$$

subject to $P_1 = P_2 = 0$, where

$$2x + y + z - 1 = P_1(x, y, z)$$

$$x - y + z - 2 = P_2(x, y, z)$$

Lagrange Multiplier Method for 2 constraints is

$$\nabla f = \lambda_1 \nabla P_1 + \lambda_2 \nabla P_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

So:

$$\left(\frac{\partial}{\partial x}\right) \quad 2x = \lambda_1(2) + \lambda_2(1) = 2\lambda_1 + \lambda_2$$

$$\left(\frac{\partial}{\partial y}\right) \quad 2y = \lambda_1(1) + \lambda_2(-1) = \lambda_1 - \lambda_2$$

$$\left(\frac{\partial}{\partial z}\right) \quad 2z = \lambda_1(-1) + \lambda_2(1) = -\lambda_1 + \lambda_2$$

plug in:

$$(P_1=0) \quad 2\lambda_1 + \lambda_2 + \frac{1}{2}(\lambda_1 - \lambda_2) - \frac{1}{2}(-\lambda_1 + \lambda_2) - 1 = 0$$

$$(P_2=0) \quad \frac{1}{2}(2\lambda_1 + \lambda_2) - \frac{1}{2}(\lambda_1 - \lambda_2) + \frac{1}{2}(-\lambda_1 + \lambda_2) - 2 = 0$$

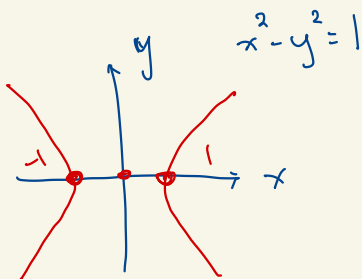
$$\Rightarrow \lambda_1 = 1/3, \lambda_2 = 4/3.$$

Sub back into $\nabla f = \lambda_1 \nabla P_1 + \lambda_2 \nabla P_2$ to get

$$x=1, \quad y=-1/2, \quad z=1/2$$

Problem 7:

a) 4.10.1



Expect: closest points
are 1, -1, so min distance
should be 1.

distance to
origin $\rightarrow$

$$D(x, y) = \sqrt{x^2 + y^2}. \quad \text{Better to minimize}$$

$$D^2(x, y) = x^2 + y^2, \quad \text{since } (\cdot)^2 \text{ is monotonic.}$$

min $D^2(x, y)$ subject to

$$0 = H(x, y) = x^2 - y^2 - 1$$

$$\nabla D^2(x,y) = \lambda \nabla H(x,y)$$

$$2x = \lambda 2x$$

$$2y = -\lambda 2y$$

• If $\lambda = 0$, then $x = 0$ and $y = 0$, but these do not satisfy $H(x,y) = 0$.

• So,

$$\lambda = 1 \Rightarrow y = 0 \text{ and } x^2 - 0^2 = 1 \text{ so } x = \pm 1$$

$$\lambda = -1 \Rightarrow x = 0 \text{ and } 0^2 - y^2 = 1, \text{ which has no solutions}$$

Two distance minimizers are $(x,y) = (1,0)$ and $(-1,0)$, as expected. Min distance = 1.

6) 4.10.8

$$\text{Maximize } T = xyz$$

$$\text{subject to } S = x^2 + y^2 + z^2 - 12 = 0$$

Notice that this is the same set of eqns as in the "box in egg" problem!

• Maximizing $T = xyz$ maximizes $8T = 8xyz$.

$$-\frac{S}{12} = \frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{12} - 1 = 0$$

$$\Rightarrow a=b=c=\sqrt{12}.$$

$$\text{So } (x,y,z) = \frac{1}{\sqrt{3}}(\sqrt{12}, \sqrt{12}, \sqrt{12}) = (2,2,2)$$

3 more:
 $(-2, -2, 2)$,
 $(-2, 2, -2)$,
 $(2, 2, -2)$.
 $(-2, 2, 2)$.
T=8

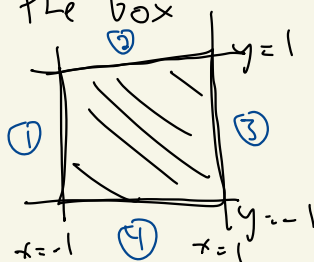
c) 4.10.11

First, try unconstrained max/min:

$$\nabla T = 0$$

$$\Rightarrow 4x - 2 = 0, \text{ so } (x, y) = (2, 0). \\ -6y = 0$$

But this is outside the box



so, optimum must lie on boundary

$$\textcircled{1} T(-1, y) = 2 - 3y^2 + 2 + 10 \\ = -3y^2 + 14$$

$$\frac{\partial T(-1, y)}{\partial y} = -6y, \text{ so } T(-1, 0) = 14 \text{ is a critical point.}$$

$$\textcircled{2} T(x, 1) = 2x^2 - 3 - 2x + 10$$

$$\frac{\partial T(x, 1)}{\partial x} = 4x - 2, \text{ so } T\left(\frac{1}{2}, 1\right) = \frac{13}{2} \text{ is a critical point}$$

$$\textcircled{3} \quad T(1, y) = 2 - 3y^2 - 2 + 10 \\ = -3y^2 + 10$$

$$\frac{\partial T(1, y)}{\partial y} = -6y, \text{ so } T(1, 0) = 12 \text{ is a critical point}$$

$$\textcircled{4} \quad T(x, -1) = 2x^2 - 3 - 2x + 10$$

$$\frac{\partial T(x, -1)}{\partial x} = 4x - 2, \text{ so } T(\frac{1}{2}, -1) = \frac{13}{2} \text{ is a critical point}$$

This leaves the corners,

$$T(1, 1) = 7$$

$$T(1, -1) = 7$$

$$T(-1, 1) = 11$$

$$T(-1, -1) = 11$$

So, the minima are at $T(\frac{1}{2}, -1) = T(\frac{1}{2}, 1) = \frac{13}{2}$

and maximum is at $T(-1, 0) = 14$.